Sensitivity analysis for Monte Carlo and Quasi Monte Carlo option pricing.

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Introduction

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Geometric Brownian Motion

If a process generate some outcome which is time-depended but can not be said ahead of time is known as a *stochastic process*.

Definition

A stochastic process $\{W(t) : 0 \leq t \leq T\}$ *is a standard Brownian motion on* $[0, T]$ *if*

1. $W(0) = 0$
2. *It has independent increments. That is, for any* $t_1, t_2, \dots, t_n$, $W(t_2) - W(t_1), W(t_3) - W(t_4) \dots, W(t_n) - W(t_{n-})$ *are independent random variables.*
3. *For every* $0 \leq s < t \leq T, W(t) - W(s) \sim \mathbb{N}(0, t-s)$

Definition

A stochastic process $\{X(t) : 0 < t < T\}$ *is said to be a general Brownian motion with a drift parameter* μ *and diffusion coefficient* σ^2 *if* $\frac{X(t)-\mu t}{\sigma}$ *is a standard Brownian motion, written as* $X(t) \sim BM(\mu, \sigma^2)$. *The general Brownian motion still follow first two properties of the standard Brownian motion. However, the third property is modified as* $X(t) - X(s) \sim \mathbb{N}(\mu(t-s), \sigma^2(t-s))$ *for any* $0 \leq s < t < T$

Geometric Brownian Motion cont

- If a stochastic process $X_t \sim BM(\mu, \sigma^2)$ where μ is the drift and σ^2 diffusion parameter, then X_t satisfies

$$dX(t) = \mu t + \sigma dW(t) \tag{1}$$

  where, $W(t)$ is the standard Brownian motion or Wiener process.

- Define $X(t) = \log S(t)$ then

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{2}$$

  is the SDE for the stock price random process.

- For a given time $t > 0$, the solution of equation (2)

$$S(t) = S(0) + \mu \int_0^t S(r)dr + \sigma \int_0^t S(r)dW(r) \tag{3}$$

Stochastic Model: Geometric Brownian Motion cont.

- A more explicit formula can be derived using Ito's formula for the function $F(\log S(t), t)$

$$dF = \left[\frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial S(t)} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial^2 S(t)}\right] dt + \left(\sigma \frac{\partial F}{\partial S(t)}\right) dW(t)$$

- After simplification we obtain

$$\begin{aligned} d \log S(t) &= \frac{1}{S(t)} dS(t) + \frac{1}{2} \frac{-1}{S^2(t)} (dS(t))^2 \\ &= \mu dt + \sigma dW(t) + \frac{1}{2} \frac{-1}{S^2(t)} (\mu S(t) dt + \sigma S(t) dW(t))^2 \\ &= (\mu - \frac{1}{2}\sigma^2) dt + \sigma dW(t) \end{aligned}$$

- For any time $t > 0$ the differential can be written as

$$S(t) = S(0) e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)} \tag{4}$$

$GBM(\mu, \sigma^2)$ Simulation

For a given time set $t_0 = 0 < t_1 < t_2 < \ldots < t_n$ the stock price $S(t)$ at time $t_0, t_1, \ldots, t_n$ can be generated by

$$S(t_{i+1}) = S(t_i) e^{(\mu - \frac{1}{2}\sigma^2)(t_{i+1} - t_i) + \sigma \sqrt{(t_{i+1} - t_i)} Z_{i+1}} \tag{5}$$

where $Z_1, Z_2, \ldots Z_n$ are independent and identically distributed standard normals and $i = \overline{0, (n-1)}$.

Sensitivity Analysis: Pathwise Derivative Method

The payoff from a European call option is defined as

$$Y = e^{-rT}(S_T - K)^+ \tag{6}$$

In pathwise derivative method we use the chain rule to compute

$$\frac{dY}{dS_0} = \frac{dY}{dS_T}\frac{dS_T}{dS_0} = e^{-rT}1_{[S_T>K]}\frac{S_T}{S_0} \tag{7}$$

That is, Delta: $\Delta = e^{-rT}1_{[S_T>K]}\frac{S_T}{S_0}$

To estimate the delta, we generate N values for S_T; say $S^1, S^2, \ldots, S^N$ and compute

$$\Delta = \frac{1}{N}\sum_{i=1}^{N} e^{-rT}1_{[S^i>K]}\frac{S^i}{S_0} \tag{8}$$

Monte-Carlo Method

For example, let say the current stock price, $S_0 = \$100.00$, Strike Price, $K = \$100.00$, interest rate, $r = 5\%$, volatility, $\sigma = 30\%$, and time interval $T = 1$.

In Monte-Carlo Method we generate sufficiently large number of the end price, S_T and take the mean of those prices. In our example, we generate 10,000 price paths.

Price Paths

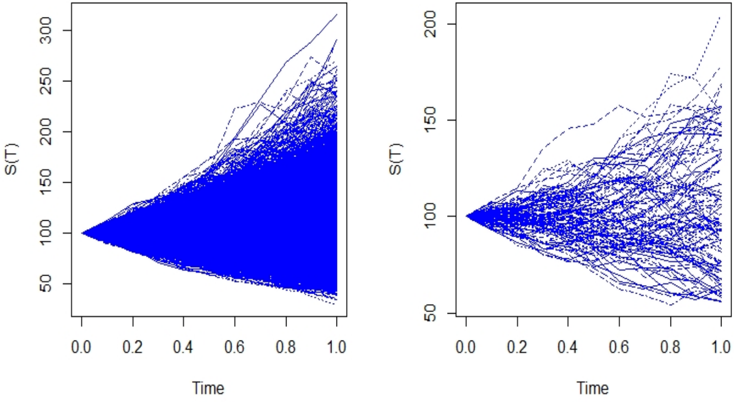

Figure: Price Paths

MC:Option Price and Sensitivity Analysis

Using the call option payoff equation (6),

$$Y = e^{-rT}(S_T - K)^+$$

and delta calculation equation (8),

$$\Delta = \frac{1}{N}\sum_{i=1}^{N} e^{-rT} 1_{[S^i > K]} \frac{S^i}{S_0}$$

we obtain

- MC Option Price=$14.05
- MC Option Delta=0.6252

Quasi-Monte Carlo Method

- We take the same example, current stock price, $S_0 = \$100.00$, Strike Price, $K = \$100.00$, interest rate, $r = 5\%$, volatility, $\sigma = 30\%$, and time interval $T = 1$.
- In Quasi-Monte Carlo Method we use low discrepancy sequence, in this case, Halton sequence to generate 10,000 price paths for the 1 time period.
- The price paths look almost same what we have seen in MC method.

QMC: Option Price and Sensitivity Analysis

Using the call option payoff equation (6),

$$Y = e^{-rT}(S_T - K)^+$$

and delta calculation equation (8),

$$\Delta = \frac{1}{N}\sum_{i=1}^{N} e^{-rT} 1_{[S^i > K]} \frac{S^i}{S_0}$$

we obtain

- QMC Option Price=$14.39
- QMC Option Delta=0.6280

Results

	Option Price	Delta
Black Scholes	\$14.23	0.6243
Monte Carlo	\$14.05	0.6252
Quasi-Monte Carlo	\$14.39	0.6280

Table: Method comparison